

Decomposition and limit theorems for a class of self-similar Gaussian processes

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Abstract

We introduce a new class of self-similar Gaussian stochastic processes, where the covariance is defined in terms of a fractional Brownian motion and another Gaussian process. A special case is the solution in time to the fractional-colored stochastic heat equation described in Tudor (2013). We prove that the process can be decomposed into a fractional Brownian motion (with a different parameter than the one that defines the covariance), and a Gaussian process first described in Lei and Nualart (2008). The component processes can be expressed as stochastic integrals with respect to the Brownian sheet. We then prove a central limit theorem about the Hermite variations of the process.

1 Introduction

The purpose of this paper is to introduce a new class of Gaussian self-similar stochastic processes related to stochastic partial differential equations, and to establish a decomposition in law and a central limit theorem for the Hermite variations of the increments of such processes.

Consider the d -dimensional stochastic heat equation

$$\frac{\partial u}{\partial t} = \frac{1}{2} \Delta u + \dot{W}, \quad t \geq 0, \quad x \in \mathbb{R}^d, \quad (1.1)$$

with zero initial condition, where \dot{W} is a zero mean Gaussian field with a covariance of the form

$$\mathbb{E} \left[\dot{W}^H(t, x) \dot{W}^H(s, y) \right] = \gamma_0(t - s) \Lambda(x - y), \quad s, t \geq 0, \quad x, y \in \mathbb{R}^d.$$

We are interested in the process $U = \{U_t, t \geq 0\}$, where $U_t = u(t, 0)$.

Suppose that \dot{W} is white in time, that is, $\gamma_0 = \delta_0$ and the spatial covariance is the Riesz kernel, that is, $\Lambda(x) = c_{d,\beta} |x|^{-\beta}$, with $\beta < \min(d, 2)$ and $c_{d,\beta} = \pi^{-d/2} 2^{\beta-d} \Gamma(\beta/2) / \Gamma((d-\beta)/2)$.

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$\beta)/2$). Then U has the covariance (see [13])

$$\mathbb{E}[U_t U_s] = D \left((t+s)^{1-\frac{\beta}{2}} - |t-s|^{1-\frac{\beta}{2}} \right), \quad s, t \geq 0, \quad (1.2)$$

for some constant

$$D = (2\pi)^{-d} (1 - \beta/2)^{-1} \int_{\mathbb{R}^d} e^{-\frac{|\xi|^2}{2}} \frac{d\xi}{|\xi|^{d-\beta}}. \quad (1.3)$$

Up to a constant, the covariance (1.2) is the covariance of the *bifractional Brownian motion* with parameters $H = \frac{1}{2}$ and $K = 1 - \frac{\beta}{2}$. We recall that, given constants $H \in (0, 1)$ and $K \in (0, 1)$, the bifractional Brownian motion $B^{H,K} = \{B_t^{H,K}, t \geq 0\}$, introduced in [3], is a centered Gaussian process with covariance

$$R_{H,K}(s, t) = \frac{1}{2^K} \left((t^{2H} + s^{2H})^K - |t-s|^{2HK} \right), \quad s, t \geq 0.$$

When $K = 1$, the process $B^H = B^{H,1}$ is simply the fractional Brownian motion (fBm) with Hurst parameter $H \in (0, 1)$, with covariance $R_H(s, t) = R_{H,1}(s, t)$. In [4], Lei and Nualart obtained the following decomposition in law for the bifractional Brownian motion

$$B^{H,K} = C_1 B^{HK} + C_2 Y_{t^{2H}}^K,$$

where B^{HK} is a fBm with Hurst parameter HK , the process Y^K is given by

$$Y_t^K = \int_0^\infty y^{-\frac{1+K}{2}} (1 - e^{-yt}) dW_y, \quad (1.4)$$

with $W = \{W_y, y \geq 0\}$ a standard Brownian motion independent of $B^{H,K}$, and C_1, C_2 are constants given by $C_1 = 2^{\frac{1-K}{2}}$ and $C_2 = \sqrt{\frac{2^{-K}}{\Gamma(1-K)}}$. The process Y^K has trajectories which are infinitely differentiable on $(0, \infty)$ and Hölder continuous of order $HK - \epsilon$ in any interval $[0, T]$ for any $\epsilon > 0$. In particular, this leads to a decomposition in law of the process U with covariance (1.2) as the sum of a fractional Brownian motion with Hurst parameter $\frac{1}{2} - \frac{\beta}{4}$ plus a regular process.

The classical one-dimensional space-time white noise can also be considered as an extension of the covariance (1.2) if we take $\beta = 1$. In this case the covariance corresponds, up to a constant, to that of a bifractional Brownian motion with parameters $H = K = \frac{1}{2}$.

The case where the noise term \dot{W} is a fractional Brownian motion with Hurst parameter $H \in (\frac{1}{2}, 1)$ in time and a spatial covariance given by the Riesz kernel, that is,

$$\mathbb{E} \left[\dot{W}^H(t, x) \dot{W}^H(s, y) \right] = \alpha_H c_{d,\beta} |s-t|^{2H-2} |x-y|^{-\beta},$$

where $0 < \beta < \min(d, 2)$ and $\alpha_H = H(2H-1)$, has been considered by Tudor and Xiao in [13]. In this case the corresponding process U has the covariance

$$\mathbb{E}[U_t U_s] = D \alpha_H \int_0^t \int_0^s |u-v|^{2H-2} (t+s-u-v)^{-\gamma} du dv. \quad (1.5)$$

where D is given in (1.3) and $\gamma = \frac{d-\beta}{2}$. This process is self-similar with parameter $H - \frac{\gamma}{2}$ and it has been studied in a series of papers [1, 7, 11, 12, 13]. In particular, in [13] it is proved that the process U can be decomposed into the sum of a scaled fBm with parameter $H - \frac{\gamma}{2}$, and a Gaussian process V with continuously differentiable trajectories. This decomposition is based on the stochastic heat equation. As a consequence, one can derive the exact uniform and local moduli of continuity and Chung-type laws of the iterated logarithm for this process. In [11], assuming that $d = 1, 2$ or 3 , a central limit theorem is obtained for the renormalized quadratic variation

$$V_n = n^{2H-\gamma-\frac{1}{2}} \sum_{j=0}^{n-1} \left\{ (U_{(j+1)T/n} - U_{jT/n})^2 - \mathbb{E}[(U_{(j+1)t/n} - U_{jT/n})^2] \right\},$$

assuming $\frac{1}{2} < H < \frac{3}{4}$, extending well-known results for fBm (see for example [5, Theorem 7.4.1]).

The purpose of this paper is to establish a decomposition in law, similar to that obtained by Lei and Nualart in [4] for the bifractional Brownian motion, and a central limit theorem for the Hermite variations of the increments, for a class of self-similar processes that includes the covariance (1.5). Consider a centered Gaussian process $\{X_t, t \geq 0\}$ with covariance

$$R(s, t) = \mathbb{E}[X_s X_t] = \mathbb{E} \left[\left(\int_0^t Z_{t-r} dB_r^H \right) \left(\int_0^s Z_{s-r} dB_r^H \right) \right], \quad (1.6)$$

where

- (i) $B^H = \{B_t^H, t \geq 0\}$ is a fBm with Hurst parameter $H \in (0, 1)$.
- (ii) $Z = \{Z_t, t > 0\}$ is a zero-mean Gaussian process, independent of B^H , with covariance

$$\mathbb{E}[Z_s Z_t] = (s + t)^{-\gamma}, \quad (1.7)$$

where $0 < \gamma < 2H$.

In other words, X is a Gaussian process with the same covariance as the process $\{\int_0^t Z_{t-r} dB_r^H, t \geq 0\}$, which is not Gaussian.

When $H \in (\frac{1}{2}, 1)$, the covariance (1.6) coincides with (1.5) with $D = 1$. However, we allow the range of parameters $0 < H < 1$ and $0 < \gamma < 2H$. In other words, up to a constant, X has the law of the solution in time of the stochastic heat equation (1.1), when $H \in (0, 1)$ and $d \geq 1$ and $\beta = d - 2\gamma$. Also of interest is that X can be constructed as a sum of stochastic integrals with respect to the Brownian sheet (see the proof of Theorem 1).

1.1 Decomposition of the process X

Our first result is the following decomposition in law of the process X as the sum of a fractional Brownian motion with Hurst parameter $\frac{\alpha}{2} = H - \frac{\gamma}{2}$ plus a process with regular trajectories.

Theorem 1. *The process X has the same law as $\{\sqrt{\kappa}B_t^{\frac{\alpha}{2}} + Y_t, t \geq 0\}$, where*

$$\kappa = \frac{1}{\Gamma(\gamma)} \int_0^\infty \frac{z^{\gamma-1}}{1+z^2} dz, \quad (1.8)$$

$B_t^{\frac{\alpha}{2}}$ is a fBm with Hurst parameter $\alpha/2$, and Y (up to a constant) has the same law as the process Y^K defined in (1.4), with $K = 2\alpha + 1$, that is, Y is a centered Gaussian process with covariance given by

$$\mathbb{E}[Y_t Y_s] = \lambda_1 \int_0^\infty y^{-\alpha-1} (1 - e^{-yt})(1 - e^{-ys}) dy,$$

where

$$\lambda_1 = \frac{4\pi}{\Gamma(\gamma)\Gamma(2H+1)\sin(\pi H)} \int_0^\infty \frac{\eta^{1-2H}}{1+\eta^2} d\eta.$$

The proof of this theorem is given in Section 3.

1.2 Hermite variations of the process

For each integer $q \geq 0$, the q th Hermite polynomial is given by

$$H_q(x) = (-1)^q e^{\frac{x^2}{2}} \frac{d^q}{dx^q} e^{-\frac{x^2}{2}}.$$

See [5, Section 1.4] for a discussion of properties of these polynomials. In particular, it is well known that the family $\{\frac{1}{q!}H_q, q \geq 0\}$ constitutes an orthonormal basis of the space $L^2(\mathbb{R}, \gamma)$, where γ is the $N(0, 1)$ measure.

Suppose $\{Z_n, n \geq 1\}$ is a stationary, Gaussian sequence, where each Z_n follows the $N(0, 1)$ distribution with covariance function $\rho(k) = \mathbb{E}[Z_n Z_{n+k}]$. If $\sum_{k=1}^\infty |\rho(k)|^q < \infty$, it is well known that as n tends to infinity, the Hermite variation

$$V_n = \frac{1}{\sqrt{n}} \sum_{j=1}^n H_q(Z_j) \quad (1.9)$$

converges in distribution to a Gaussian random variable with mean zero and variance given by $\sigma^2 = \sum_{k=1}^\infty \rho(k)^q$. This result was proved by Breuer and Major in [2]. In particular, if B^H is a fBm, then the sequence $\{Z_{j,n}, 0 \leq j \leq n-1\}$ defined by

$$Z_{j,n} = n^H \left(B_{\frac{j+1}{n}}^H - B_{\frac{j}{n}}^H \right)$$

is a stationary sequence with unit variance. As a consequence, $H < 1 - \frac{1}{q}$, we have that

$$\frac{1}{\sqrt{n}} \sum_{j=0}^{n-1} H_q \left(n^H \left(B_{\frac{j+1}{n}}^H - B_{\frac{j}{n}}^H \right) \right)$$

converges to a normal law with variance given by

$$\sigma_q^2 = \frac{q!}{2^q} \sum_{m \in \mathbb{Z}} (|m+1|^{2H} - 2|m|^{2H} + |m-1|^{2H})^q. \quad (1.10)$$

See [2] and Theorem 7.4.1 of [5].

The above Breuer-Major theorem can not be applied to our process because X is not necessarily stationary. However, we have a comparable result.

Theorem 2. *Let $q \geq 2$ be an integer and fix a real $T > 0$. Suppose that $\alpha < 2 - \frac{1}{q}$. For $t \in [0, T]$, define,*

$$F_n(t) = n^{-\frac{1}{2}} \sum_{j=0}^{\lfloor nt \rfloor - 1} H_q \left(\frac{\Delta X_{\frac{j}{n}}}{\left\| \Delta X_{\frac{j}{n}} \right\|_{L^2(\Omega)}} \right),$$

where $H_q(x)$ denotes the q th Hermite polynomial. Then as $n \rightarrow \infty$, the stochastic process $\{F_n(t), t \in [0, T]\}$ converges in law in the Skorohod space $D([0, T])$, to a scaled Brownian motion $\{\sigma B_t, t \in [0, T]\}$, where $\{B_t, t \in [0, T]\}$ is a standard Brownian motion and $\sigma = \sqrt{\sigma^2}$ is given by

$$\sigma^2 = \frac{q!}{2^q} \sum_{m \in \mathbb{Z}} (|m+1|^\alpha - 2|m|^\alpha + |m-1|^\alpha)^q. \quad (1.11)$$

The proof of this theorem is given in Section 4.

2 Preliminaries

2.1 Analysis on the Wiener space

The reader may refer to [5, 6] for a detailed coverage of this topic. Let $Z = \{Z(h), h \in \mathcal{H}\}$ be an *isonormal Gaussian process* on a probability space (Ω, \mathcal{F}, P) , indexed by a real separable Hilbert space \mathcal{H} . This means that Z is a family of Gaussian random variables such that $\mathbb{E}[Z(h)] = 0$ and $\mathbb{E}[Z(h)Z(g)] = \langle h, g \rangle_{\mathcal{H}}$ for all $h, g \in \mathcal{H}$.

For integers $q \geq 1$, let $\mathcal{H}^{\otimes q}$ denote the q th tensor product of \mathcal{H} , and $\mathcal{H}^{\odot q}$ denote the subspace of symmetric elements of $\mathcal{H}^{\otimes q}$.

Let $\{e_n, n \geq 1\}$ be a complete orthonormal system in \mathcal{H} . For elements $f, g \in \mathcal{H}^{\odot q}$ and $p \in \{0, \dots, q\}$, we define the p th-order contraction of f and g as that element of $\mathcal{H}^{\otimes 2(q-p)}$ given by

$$f \otimes_p g = \sum_{i_1, \dots, i_p=1}^{\infty} \langle f, e_{i_1} \otimes \dots \otimes e_{i_p} \rangle_{\mathcal{H}^{\otimes p}} \otimes \langle g, e_{i_1} \otimes \dots \otimes e_{i_p} \rangle_{\mathcal{H}^{\otimes p}}, \quad (2.1)$$

where $f \otimes_0 g = f \otimes g$. Note that, if $f, g \in \mathcal{H}^{\odot q}$, then $f \otimes_q g = \langle f, g \rangle_{\mathcal{H}^{\odot q}}$. In particular, if f, g are real-valued functions in $\mathcal{H}^{\otimes q} = L^2(\mathbb{R}^2, \mathcal{B}^2, \mu)$ for a non-atomic measure μ , then we have

$$f \otimes_1 g = \int_{\mathbb{R}} f(s, t_1) g(s, t_2) \mu(ds). \quad (2.2)$$

Let \mathcal{H}_q be the q th Wiener chaos of Z , that is, the closed linear subspace of $L^2(\Omega)$ generated by the random variables $\{H_q(Z(h)), h \in \mathcal{H}, \|h\|_{\mathcal{H}} = 1\}$, where $H_q(x)$ is the q th Hermite polynomial. It can be shown (see [5, Proposition 2.2.1]) that if $Z, Y \sim N(0, 1)$ are jointly Gaussian, then

$$\mathbb{E}[H_p(Z)H_q(Y)] = \begin{cases} p! (\mathbb{E}[ZY])^p & \text{if } p = q \\ 0 & \text{otherwise} \end{cases}. \quad (2.3)$$

For $q \geq 1$, it is known that the map

$$I_q(h^{\otimes q}) = H_q(Z(h)) \quad (2.4)$$

provides a linear isometry between $\mathcal{H}^{\odot q}$ (equipped with the modified norm $\sqrt{q!} \cdot \|\cdot\|_{\mathcal{H}^{\otimes q}}$) and \mathcal{H}_q , where $I_q(\cdot)$ is the generalized Wiener-Itô stochastic integral (see [5, Theorem 2.7.7]). By convention, $\mathcal{H}_0 = \mathbb{R}$ and $I_0(x) = x$.

We use the following integral multiplication theorem from [6, Proposition 1.1.3]. Suppose $f \in \mathcal{H}^{\odot p}$ and $g \in \mathcal{H}^{\odot q}$. Then

$$I_p(f)I_q(g) = \sum_{r=0}^{p \wedge q} r! \binom{p}{r} \binom{q}{r} I_{p+q-2r}(f \widetilde{\otimes}_r g), \quad (2.5)$$

where $f \widetilde{\otimes}_r g$ denotes the symmetrization of $f \otimes_r g$. For a product of more than two integrals, see Peccati and Taqqu [8].

2.2 Stochastic integration and fBm

We refer to the ‘time domain’ and ‘spectral domain’ representations of fBm. The reader may refer to [9, 10] for details. Let \mathcal{E} denote the set of real-valued step functions on \mathbb{R} . Let B^H denote fBm with Hurst parameter H . For this case, we view B^H as an isonormal Gaussian process on the Hilbert space \mathfrak{H} , which is the closure of \mathcal{E} with respect to the inner product $\langle f, g \rangle_{\mathfrak{H}} = \mathbb{E}[I(f)I(g)]$. Consider also the inner product space

$$\tilde{\Lambda}_H = \left\{ f : f \in L^2(\mathbb{R}), \int_{\mathbb{R}} |\mathcal{F}f(\xi)|^2 |\xi|^{1-2H} d\xi < \infty \right\},$$

where $\mathcal{F}f = \int_{\mathbb{R}} f(x)e^{i\xi x} dx$ is the Fourier transform, and the inner product of $\tilde{\Lambda}_H$ is given by

$$\langle f, g \rangle_{\tilde{\Lambda}_H} = \frac{1}{C_H^2} \int_{\mathbb{R}} \mathcal{F}f(\xi) \overline{\mathcal{F}g(\xi)} |\xi|^{1-2H} d\xi, \quad (2.6)$$

where $C_H = \left(\frac{2\pi}{\Gamma(2H+1) \sin(\pi H)} \right)^{\frac{1}{2}}$. It is known (see [9, Theorem 3.1]) that the space $\tilde{\Lambda}_H$ is isometric to a subspace of \mathfrak{H} , and $\tilde{\Lambda}_H$ contains \mathcal{E} as a dense subset. This inner product (2.6) is known as the ‘spectral measure’ of fBm. In the case $H \in (\frac{1}{2}, 1)$, there is another isometry from the space

$$|\Lambda_H| = \left\{ f : \int_0^\infty \int_0^\infty |f(u)||f(v)||u-v|^{2H-2} du dv < \infty \right\}$$

to a subspace of \mathfrak{H} , where the inner product is defined as

$$\langle f, g \rangle_{|\Lambda_H|} = H(2H-1) \int_0^\infty \int_0^\infty f(u)g(v)|u-v|^{2H-2} du dv,$$

see [9] or [6, Section 5.1].

3 Proof of Theorem 1

For any $\gamma > 0$ and $\lambda > 0$, we can write

$$\lambda^{-\gamma} = \frac{1}{\Gamma(\gamma)} \int_0^\infty y^{\gamma-1} e^{-\lambda y} dy,$$

where Γ is the Gamma function defined by $\Gamma(\gamma) = \int_0^\infty y^{\gamma-1} e^{-y} dy$. As a consequence, the covariance (1.7) can be written as

$$\mathbb{E}[Z_s Z_t] = \frac{1}{\Gamma(\gamma)} \int_0^\infty y^{\gamma-1} e^{-(t+s)y} dy. \quad (3.1)$$

Notice that this representation implies the covariance (1.7) is positive definite. Taking first the expectation with respect to the process Z , and using formula (3.1), we obtain

$$\begin{aligned} R(s, t) &= \frac{1}{\Gamma(\gamma)} \int_0^\infty \mathbb{E} \left[\left(\int_0^t e^{yu} dB_u^H \right) \left(\int_0^t e^{yv} dB_v^H \right) \right] y^{\gamma-1} e^{-(t+s)y} dy \\ &= \frac{1}{\Gamma(\gamma)} \int_0^\infty \langle e^{yu} \mathbf{1}_{[0,t]}(u), e^{yv} \mathbf{1}_{[0,s]}(v) \rangle_{\mathfrak{H}} y^{\gamma-1} e^{-(t+s)y} dy. \end{aligned}$$

Using the isometry between $\tilde{\Lambda}_H$ and a subspace of \mathfrak{H} (see section 2.2), we can write

$$\begin{aligned} \langle e^{yu} \mathbf{1}_{[0,t]}(u), e^{yv} \mathbf{1}_{[0,s]}(v) \rangle_{\mathfrak{H}} &= C_H^{-2} \int_{\mathbb{R}} |\xi|^{1-2H} (\mathcal{F} \mathbf{1}_{[0,t]} e^{y\cdot}) (\overline{\mathcal{F} \mathbf{1}_{[0,s]} e^{y\cdot}}) d\xi \\ &= C_H^{-2} \int_{\mathbb{R}} \frac{|\xi|^{1-2H}}{y^2 + \xi^2} (e^{yt+i\xi t} - 1) (e^{ys-i\xi s} - 1) d\xi, \end{aligned}$$

where $(\mathcal{F} \mathbf{1}_{[0,t]} e^{y\cdot})$ denotes the Fourier transform and $C_H = \left(\frac{2\pi}{\Gamma(2H+1) \sin(\pi H)} \right)^{\frac{1}{2}}$. This allows us to write, making the change of variable $\xi = \eta y$,

$$\begin{aligned} R(s, t) &= \frac{1}{\Gamma(\gamma) C_H^2} \int_0^\infty \int_{\mathbb{R}} y^{\gamma-1} \frac{|\xi|^{1-2H}}{y^2 + \xi^2} (e^{i\xi t} - e^{-yt}) (e^{-i\xi s} - e^{-ys}) d\xi dy \\ &= \frac{1}{\Gamma(\gamma) C_H^2} \int_0^\infty \int_{\mathbb{R}} y^{-\alpha-1} \frac{|\eta|^{1-2H}}{1 + \eta^2} (e^{i\eta y t} - e^{-yt}) (e^{-i\eta y s} - e^{-ys}) d\eta dy, \end{aligned} \quad (3.2)$$

where $\alpha = 2H - \gamma$. By Euler's identity, adding and subtracting 1 to compensate the singularity of $y^{-\alpha-1}$ at the origin, we can write

$$e^{i\eta y t} - e^{-yt} = (\cos(\eta y t) - 1 + i \sin(\eta y t)) + (1 - e^{-yt}). \quad (3.3)$$

Substituting (3.3) into (3.2) and taking into account that the integral of the imaginary part vanishes because it is an odd function, we obtain

$$\begin{aligned} R(s, t) = & \frac{2}{\Gamma(\gamma)C_H^2} \int_0^\infty \int_0^\infty y^{-\alpha-1} \frac{\eta^{1-2H}}{1+\eta^2} \left((1 - \cos(\eta yt))(1 - \cos(\eta ys)) \right. \\ & + \sin(\eta yt) \sin(\eta ys) + (\cos(\eta ys) - 1)(1 - e^{-yt}) + (\cos(\eta yt) - 1)(1 - e^{-ys}) \\ & \left. + (1 - e^{-yt})(1 - e^{-ys}) \right) d\eta dy. \end{aligned}$$

Let $B^{(j)} = \{B^{(j)}(\eta, t), \eta \geq 0, t \geq 0\}$, $j = 1, 2$ denote two independent Brownian sheets. That is, for $j = 1, 2$, $B^{(j)}$ is a continuous Gaussian field with mean zero and covariance given by

$$\mathbb{E} [B^{(j)}(\eta, t) B^{(j)}(\xi, s)] = \min(\eta, \xi) \times \min(t, s).$$

We define the the following stochastic processes:

$$U_t = \frac{\sqrt{2}}{\sqrt{\Gamma(\gamma)}C_H} \int_0^\infty \int_0^\infty y^{-\frac{\alpha}{2}-\frac{1}{2}} \sqrt{\frac{\eta^{1-2H}}{1+\eta^2}} (\cos(\eta yt) - 1) B^{(1)}(d\eta, dy), \quad (3.4)$$

$$V_t = \frac{\sqrt{2}}{\sqrt{\Gamma(\gamma)}C_H} \int_0^\infty \int_0^\infty y^{-\frac{\alpha}{2}-\frac{1}{2}} \sqrt{\frac{\eta^{1-2H}}{1+\eta^2}} (\sin(\eta yt)) B^{(2)}(d\eta, dy), \quad (3.5)$$

$$Y_t = \frac{\sqrt{2}}{\sqrt{\Gamma(\gamma)}C_H} \int_0^\infty \int_0^\infty y^{-\frac{\alpha}{2}-\frac{1}{2}} \sqrt{\frac{\eta^{1-2H}}{1+\eta^2}} (1 - e^{-yt}) B^{(1)}(d\eta, dy), \quad (3.6)$$

where the integrals are Wiener-Itô integrals with respect to the Brownian sheet. We then define the stochastic process $X = \{X_t, t \geq 0\}$ by $X_t = U_t + V_t + Y_t$, and we have $\mathbb{E}[X_s X_t] = R(s, t)$ as given in (3.2). These processes have the following properties:

(I) The process $W_t = U_t + V_t$ is a fractional Brownian motion with Hurst parameter $\frac{\alpha}{2}$ scaled with the constant $\sqrt{\kappa}$. In fact, the covariance of this process is

$$\begin{aligned} \mathbb{E}[W_t W_s] &= \frac{2}{\Gamma(\gamma)C_H^2} \int_0^\infty \int_0^\infty y^{-\alpha-1} \frac{\eta^{1-2H}}{1+\eta^2} \left((\cos(\eta yt) - 1)(\cos(\eta ys) - 1) \right. \\ &\quad \left. + \sin(\eta yt) \sin(\eta ys) \right) d\eta dy \\ &= \frac{1}{\Gamma(\gamma)C_H^2} \int_0^\infty \int_{\mathbb{R}} y^{\gamma-1} \frac{|\xi|^{1-2H}}{y^2 + \xi^2} (e^{i\xi t} - 1)(e^{-i\xi s} - 1) d\xi dy. \end{aligned}$$

Integrating in the variable y we finally obtain

$$\mathbb{E}[W_t W_s] = \frac{c_1}{\Gamma(\gamma)C_H^2} \int_{\mathbb{R}} \frac{(e^{i\xi t} - 1)(e^{-i\xi s} - 1)}{|\xi|^{\alpha+1}} d\xi,$$

where $c_1 = \int_0^\infty \frac{z^{\gamma-1}}{1+z^2} dz = \kappa \Gamma(\gamma)$. Taking into account the Fourier transform representation of fBm (see [10, page 328]), this implies $\kappa^{-\frac{1}{2}} W$ is a fractional Brownian motion with Hurst parameter $\frac{\alpha}{2}$.

(II) The process Y coincides, up to a constant, with the process Y^K introduced in (1.4) with $K = 2\alpha + 1$. In fact, the covariance of this process is given by

$$\mathbb{E}[Y_t Y_s] = \frac{2c_2}{\Gamma(\gamma)C_H^2} \int_0^\infty y^{-\alpha-1} (1 - e^{-yt})(1 - e^{-ys}) dy, \quad (3.7)$$

where

$$c_2 = \int_0^\infty \frac{\eta^{1-2H}}{1 + \eta^2} d\eta.$$

Notice that the process X is self-similar with exponent $\frac{\alpha}{2}$. This concludes the proof of Theorem 1.

4 Proof of Theorem 2

Along the proof, the symbol C denotes a generic, positive constant, which may change from line to line. The value of C will depend on parameters of the process and on T , but not on the increment width n^{-1} .

For integers $n \geq 1$, define a partition of $[0, \infty)$ composed of the intervals $\{[\frac{j}{n}, \frac{j+1}{n}), j \geq 0\}$. For the process X and related processes U, V, W, Y defined in Section 3, we introduce the notation

$$\Delta X_{\frac{j}{n}} = X_{\frac{j+1}{n}} - X_{\frac{j}{n}} \text{ and } \Delta X_0 = X_{\frac{1}{n}},$$

with corresponding notation for U, V, W, Y . We start the proof of Theorem 2 with two technical results about the components of the increments.

4.1 Preliminary Lemmas

Lemma 3. *Using above notation with integers $n \geq 2$ and $j, k \geq 0$, we have*

(a) $\mathbb{E} \left[\Delta W_{\frac{j}{n}} \Delta W_{\frac{k}{n}} \right] = \frac{\kappa}{2} n^{-\alpha} (|j - k - 1|^\alpha - 2|j - k|^\alpha + |j - k + 1|^\alpha)$, where κ is defined in (1.8).

(b) For $j + k \geq 1$,

$$\left| \mathbb{E} \left[\Delta Y_{\frac{j}{n}} \Delta Y_{\frac{k}{n}} \right] \right| \leq C n^{-\alpha} (j + k)^{\alpha-2}$$

for a constant $C > 0$ that is independent of j, k and n .

Proof. Property (a) is well-known for fractional Brownian motion. For (b), we have from (3.7):

$$\begin{aligned} \mathbb{E} \left[\Delta Y_{\frac{j}{n}} \Delta Y_{\frac{k}{n}} \right] &= \frac{2c_2}{\Gamma(\gamma)C_H^2 n^\alpha} \int_0^\infty y^{-\alpha-1} (e^{-yj} - e^{-y(j+1)}) (e^{-yk} - e^{-y(k+1)}) dy \\ &= \frac{2c_2}{\Gamma(\gamma)C_H^2 n^\alpha} \int_0^\infty y^{-\alpha+1} \int_0^1 \int_0^1 e^{-y(j+k+u+v)} du dv dy. \end{aligned}$$

Note that the above integral is nonnegative, and we can bound this with

$$\begin{aligned} \left| \mathbb{E} \left[\Delta Y_{\frac{j}{n}} \Delta Y_{\frac{k}{n}} \right] \right| &\leq C n^{-\alpha} \int_0^\infty y^{-\alpha+1} e^{-y(j+k)} dy \\ &= C n^{-\alpha} (j+k)^{\alpha-2} \int_0^\infty u^{-\alpha+1} e^{-u} du \\ &\leq C n^{-\alpha} (j+k)^{\alpha-2}. \end{aligned}$$

□

Lemma 4. For $n \geq 2$ fixed and integers $j, k \geq 1$,

$$\left| \mathbb{E} \left[\Delta W_{\frac{j}{n}} \Delta Y_{\frac{k}{n}} \right] \right| \leq C n^{-\alpha} j^{2H-2} k^{-\gamma}$$

for a constant $C > 0$ that is independent of j, k and n .

Proof. From (3.4) - (3.6) in the proof of Theorem 1, observe that

$$\mathbb{E} \left[\Delta W_{\frac{j}{n}} \Delta Y_{\frac{k}{n}} \right] = \mathbb{E} \left[(\Delta U_{\frac{j}{n}} + \Delta V_{\frac{j}{n}}) \Delta Y_{\frac{k}{n}} \right] = \mathbb{E} \left[\Delta U_{\frac{j}{n}} \Delta Y_{\frac{k}{n}} \right].$$

Assume $s, t > 0$. By self-similarity we can define the covariance function ψ by $\mathbb{E} [U_t Y_s] = s^\alpha \mathbb{E} [U_{t/s} Y_1] = s^\alpha \psi(t/s)$, where, using the change-of-variable $\theta = \eta x$,

$$\begin{aligned} \psi(x) &= \int_0^\infty \int_0^\infty y^{-\alpha-1} \frac{\eta^{1-2H}}{1+\eta^2} (\cos(y\eta x) - 1) (1 - e^{-y}) d\eta dy \\ &= \int_0^\infty y^{-\alpha-1} (1 - e^{-y}) \int_0^\infty \frac{\theta^{1-2H} x^{2H}}{x^2 + \theta^2} (\cos(y\theta) - 1) d\theta dy. \end{aligned}$$

Then using the fact that

$$\left| \frac{\theta^{1-2H} x^{2H}}{x^2 + \theta^2} \right| \leq |\theta^{-2H}| |x|^{2H-1}, \quad (4.1)$$

we see that $|\psi(x)| \leq C x^{2H-1}$, and

$$\begin{aligned} \psi'(x) &= 2H \int_0^\infty y^{-\alpha-1} (1 - e^{-y}) \int_0^\infty \frac{\theta^{1-2H} x^{2H-1}}{x^2 + \theta^2} (\cos(y\theta) - 1) d\theta dy \\ &\quad - 2 \int_0^\infty y^{-\alpha-1} (1 - e^{-y}) \int_0^\infty \frac{\theta^{1-2H} x^{2H+1}}{(x^2 + \theta^2)^2} (\cos(y\theta) - 1) d\theta dy. \end{aligned}$$

Using (4.1) and similarly

$$\left| \frac{\theta^{1-2H} x^{2H+1}}{(x^2 + \theta^2)^2} \right| \leq |\theta^{-2H}| |x|^{2H-2}, \quad (4.2)$$

we can write

$$|\psi'(x)| \leq x^{2H-2} |2H - 2| \int_0^\infty y^{-\alpha-1} (1 - e^{-y}) \int_0^\infty \theta^{-2H} (\cos(y\theta) - 1) d\theta dy \leq C x^{2H-2}.$$

By continuing the computation, we can find that $|\psi''(x)| \leq Cx^{2H-3}$. We have for $j, k \geq 1$,

$$\begin{aligned} \mathbb{E} \left[\Delta U_{\frac{j}{n}} \Delta Y_{\frac{k}{n}} \right] &= n^{-\alpha} (k+1)^\alpha \left(\psi \left(\frac{j+1}{k+1} \right) - \psi \left(\frac{j}{k+1} \right) \right) \\ &\quad - n^{-\alpha} k^\alpha \left(\psi \left(\frac{j+1}{k} \right) - \psi \left(\frac{j}{k} \right) \right) \\ &= n^{-\alpha} ((k+1)^\alpha - k^\alpha) \left(\psi \left(\frac{j+1}{k+1} \right) - \psi \left(\frac{j}{k+1} \right) \right) \\ &\quad + n^{-\alpha} k^\alpha \left(\psi \left(\frac{j+1}{k+1} \right) - \psi \left(\frac{j}{k+1} \right) - \psi \left(\frac{j+1}{k} \right) + \psi \left(\frac{j}{k} \right) \right). \end{aligned}$$

With the above bounds on ψ and its derivatives, the first term is bounded by

$$\begin{aligned} n^{-\alpha} |(k+1)^\alpha - k^\alpha| \left| \psi \left(\frac{j+1}{k+1} \right) - \psi \left(\frac{j}{k+1} \right) \right| \\ \leq \alpha n^{-\alpha} \int_0^1 (k+u)^{\alpha-1} du \int_0^{\frac{1}{k+1}} \left| \psi' \left(\frac{j}{k+1} + v \right) \right| dv \\ \leq C n^{-\alpha} k^{\alpha-2} \left(\frac{j}{k} \right)^{2H-2} \leq C n^{-\alpha} k^{-\gamma} j^{2H-2}, \end{aligned}$$

and

$$\begin{aligned} n^{-\alpha} k^\alpha \left| \psi \left(\frac{j+1}{k+1} \right) - \psi \left(\frac{j}{k+1} \right) - \psi \left(\frac{j+1}{k} \right) + \psi \left(\frac{j}{k} \right) \right| \\ = n^{-\alpha} k^\alpha \left| \int_0^{\frac{1}{k+1}} \psi' \left(\frac{j}{k+1} + u \right) du - \int_0^{\frac{1}{k}} \psi' \left(\frac{j}{k} + u \right) du \right| \\ \leq n^{-\alpha} k^\alpha \int_{\frac{1}{k+1}}^{\frac{1}{k}} \left| \psi' \left(\frac{j}{k} + u \right) \right| du + \int_0^{\frac{1}{k+1}} \int_{\frac{j}{k+1}}^{\frac{j}{k}} |\psi''(u+v)| dv du \\ \leq C n^{-\alpha} k^{\alpha-2} \left(\frac{j}{k} \right)^{2H-2} + C n^{-\alpha} k^{\alpha-3} j \left(\frac{j}{k} \right)^{2H-3} \leq C n^{-\alpha} k^{-\gamma} j^{2H-2}. \end{aligned}$$

This concludes the proof of the lemma. \square

4.2 Proof of Theorem 2

We will make use of the notation $\beta_{j,n} = \left\| \Delta X_{\frac{j}{n}} \right\|_{L^2(\Omega)}$. From Lemma 3 and Lemma 4 we have

$$\beta_{j,n}^2 = \kappa n^{-\alpha} (1 + \theta_{j,n}),$$

where $|\theta_{j,n}| \leq Cj^{\alpha-2}$ if $j \geq 1$. Notice that, in the definition of $F_n(t)$, it suffices to consider the sum for $j \geq n_0$ for a fixed n_0 . Then, we can choose n_0 in such a way that $Cn_0^{\alpha-2} \leq \frac{1}{2}$, which implies

$$\beta_{j,n}^2 \geq \kappa n^{-\alpha} (1 - Cj^{\alpha-2}) \quad (4.3)$$

for any $j \geq n_0$.

By (2.4),

$$\beta_{j,n}^q H_q \left(\beta_{j,n}^{-1} \Delta X_{\frac{j}{n}} \right) = I_q^X \left(\left(\mathbf{1}_{\left[\frac{j}{n}, \frac{j+1}{n}\right)} \right)^{\otimes q} \right),$$

where I_q^X denotes the multiple stochastic integral of order q with respect to the process X . Thus, we can write

$$F_n(t) = n^{-\frac{1}{2}} \sum_{j=n_0}^{\lfloor nt \rfloor - 1} \beta_{j,n}^{-q} I_q^X \left(\mathbf{1}_{\left[\frac{j}{n}, \frac{j+1}{n}\right]}^{\otimes q} \right).$$

The decomposition $X = W + Y$ leads to

$$I_q^X \left(\mathbf{1}_{\left[\frac{j}{n}, \frac{j+1}{n}\right]}^{\otimes q} \right) = \sum_{r=0}^q \binom{q}{r} I_r^W \left(\mathbf{1}_{\left[\frac{j}{n}, \frac{j+1}{n}\right]}^{\otimes r} \right) I_{q-r}^Y \left(\mathbf{1}_{\left[\frac{j}{n}, \frac{j+1}{n}\right]}^{\otimes q-r} \right).$$

We are going to show that the terms with $r = 0, \dots, q-1$ do not contribute to the limit. Define

$$G_n(t) = n^{-\frac{1}{2}} \sum_{j=n_0}^{\lfloor nt \rfloor - 1} \beta_{j,n}^{-q} I_q^W \left(\mathbf{1}_{\left[\frac{j}{n}, \frac{j+1}{n}\right]}^{\otimes q} \right)$$

and

$$\tilde{G}_n(t) = n^{-\frac{1}{2}} \sum_{j=n_0}^{\lfloor nt \rfloor - 1} \|\Delta W_{j/n}\|_{L^2(\Omega)}^{-q} I_q^W \left(\mathbf{1}_{\left[\frac{j}{n}, \frac{j+1}{n}\right]}^{\otimes q} \right).$$

Consider the decomposition

$$F_n(t) = (F_n(t) - G_n(t)) + (G_n(t) - \tilde{G}_n(t)) + \tilde{G}_n(t).$$

Notice that all these processes vanish at $t = 0$. We claim that for any $0 \leq s < t \leq T$, we have

$$\mathbb{E}[|F_n(t) - G_n(t) - (F_n(s) - G_n(s))|^2] \leq \frac{(\lfloor nt \rfloor - \lfloor ns \rfloor)^\delta}{n} \quad (4.4)$$

and

$$\mathbb{E}[|G_n(t) - \tilde{G}_n(t) - (G_n(s) - \tilde{G}_n(s))|^2] \leq \frac{(\lfloor nt \rfloor - \lfloor ns \rfloor)^\delta}{n}, \quad (4.5)$$

where $0 \leq \delta < 1$. By Lemma 3, $\|\Delta W_{j/n}\|_{L^2(\Omega)}^2 = \kappa n^{-\alpha}$ for every j . As a consequence, using (2.4) we can also write

$$\tilde{G}_n(t) = n^{-\frac{1}{2}} \sum_{j=n_0}^{\lfloor nt \rfloor - 1} H_q \left(\kappa^{-\frac{1}{2}} n^{\frac{\alpha}{2}} \Delta W_{\frac{j}{n}} \right).$$

Since $\kappa^{-\frac{1}{2}} W$ is a fractional Brownian motion, the Breuer-Major theorem implies that the process \tilde{G} converges in $D([0, T])$ to a scaled Brownian motion $\{\sigma B_t, t \in [0, T]\}$, where

σ^2 is given in (1.11). By the fact that all the p -norms are equivalent on a fixed Wiener chaos, the estimates (4.4) and (4.5) lead to

$$\mathbb{E}[|F_n(t) - G_n(t) - (F_n(s) - G_n(s))|^{2p}] \leq \frac{(\lfloor nt \rfloor - \lfloor ns \rfloor)^{\delta p}}{n^p} \quad (4.6)$$

and

$$\mathbb{E}[|G_n(t) - \tilde{G}_n(t) - (G_n(s) - \tilde{G}_n(s))|^{2p}] \leq \frac{(\lfloor nt \rfloor - \lfloor ns \rfloor)^{\delta p}}{n^p}, \quad (4.7)$$

for all $p \geq 1$. Letting n tend to infinity, we deduce from (4.6) and (4.7) that the sequences $F_n - G_n$ and $G_n - \tilde{G}_n$ converge to zero in the topology of $D([0, T])$, as n tends to infinity.

Proof of (4.4): We can write

$$\mathbb{E}[|F_n(t) - G_n(t) - (F_n(s) - G_n(s))|^2] \leq C \sum_{r=0}^{q-1} \mathbb{E}[\Phi_{r,n}^2],$$

where

$$\Phi_{r,n} = n^{-\frac{1}{2}} \sum_{j=\lfloor ns \rfloor \vee n_0}^{\lfloor nt \rfloor - 1} \beta_{j,n}^{-q} I_r^W \left(\mathbf{1}_{\left[\frac{j}{n}, \frac{j+1}{n}\right)}^{\otimes r} \right) I_{q-r}^Y \left(\mathbf{1}_{\left[\frac{j}{n}, \frac{j+1}{n}\right)}^{\otimes q-r} \right).$$

We have, using (4.3),

$$\begin{aligned} \mathbb{E}[\Phi_{r,n}^2] &\leq n^{-1+q\alpha} \\ &\times \sum_{j,k=\lfloor ns \rfloor \vee n_0}^{\lfloor nt \rfloor - 1} \left| \mathbb{E} \left[I_r^W \left(\mathbf{1}_{\left[\frac{j}{n}, \frac{j+1}{n}\right)}^{\otimes r} \right) I_{q-r}^Y \left(\mathbf{1}_{\left[\frac{j}{n}, \frac{j+1}{n}\right)}^{\otimes q-r} \right) I_r^W \left(\mathbf{1}_{\left[\frac{k}{n}, \frac{k+1}{n}\right)}^{\otimes r} \right) I_{q-r}^Y \left(\mathbf{1}_{\left[\frac{k}{n}, \frac{k+1}{n}\right)}^{\otimes q-r} \right) \right] \right|. \end{aligned}$$

Using a diagram method for the expectation of four stochastic integrals (see [8]), we find that, for any j, k , the above expectation consists of a sum of terms of the form

$$\left(\mathbb{E} \left[\Delta W_{\frac{j}{n}} \Delta W_{\frac{k}{n}} \right] \right)^{a_1} \left(\mathbb{E} \left[\Delta Y_{\frac{j}{n}} \Delta Y_{\frac{k}{n}} \right] \right)^{a_2} \left(\mathbb{E} \left[\Delta W_{\frac{j}{n}} \Delta Y_{\frac{k}{n}} \right] \right)^{a_3} \left(\mathbb{E} \left[\Delta Y_{\frac{j}{n}} \Delta W_{\frac{k}{n}} \right] \right)^{a_4},$$

where the a_i are nonnegative integers such that $a_1 + a_2 + a_3 + a_4 = q$, $a_1 \leq r \leq q-1$, and $a_2 \leq q-r$. First, consider the case with $a_3 = a_4 = 0$, so that we have the sum

$$n^{-1+q\alpha} \sum_{j,k=\lfloor ns \rfloor \vee n_0}^{\lfloor nt \rfloor - 1} \left(\mathbb{E} \left[\Delta W_{\frac{j}{n}} \Delta W_{\frac{k}{n}} \right] \right)^{a_1} \left(\mathbb{E} \left[\Delta Y_{\frac{j}{n}} \Delta Y_{\frac{k}{n}} \right] \right)^{q-a_1},$$

where $0 \leq a_1 \leq q-1$. Applying Lemma 3, we can control each of the terms in the above sum by

$$n^{-q\alpha} (|j-k+1|^\alpha - 2|j-k|^\alpha + |j-k-1|^\alpha)^{a_1} (j+k)^{(q-a_1)(\alpha-2)},$$

which gives

$$\begin{aligned}
& n^{-1+q\alpha} \sum_{j,k=\lfloor ns \rfloor \vee n_0}^{\lfloor nt \rfloor - 1} \left| \mathbb{E} \left[\Delta W_{\frac{j}{n}} \Delta W_{\frac{k}{n}} \right] \right|^{a_1} \left| \mathbb{E} \left[\Delta Y_{\frac{j}{n}} \Delta Y_{\frac{k}{n}} \right] \right|^{q-a_1} \\
& \leq C n^{-1} \left(\sum_{j=\lfloor ns \rfloor \vee n_0}^{\lfloor nt \rfloor - 1} j^{\alpha-2} + \sum_{j,k=\lfloor ns \rfloor \vee n_0, j \neq k}^{\lfloor nt \rfloor - 1} |j-k|^{(q-1)(\alpha-2)} (j+k)^{\alpha-2} \right) \\
& \leq C n^{-1} \sum_{j=\lfloor ns \rfloor \vee n_0}^{\lfloor nt \rfloor - 1} (j^{\alpha-2} + j^{q(\alpha-2)+1}) \\
& \leq C n^{-1} (\lfloor nt \rfloor - \lfloor ns \rfloor)^{(\alpha-1)\vee 0} + (\lfloor nt \rfloor - \lfloor ns \rfloor)^{[q(\alpha-2)+2]\vee 0}. \tag{4.8}
\end{aligned}$$

Next, we consider the case where $a_3 + a_4 \geq 1$. By Lemma 3, we have that, up to a constant C ,

$$\left| \mathbb{E} \left[\Delta Y_{\frac{j}{n}} \Delta Y_{\frac{k}{n}} \right] \right| \leq C \left| \mathbb{E} \left[\Delta W_{\frac{j}{n}} \Delta W_{\frac{k}{n}} \right] \right|,$$

so we may assume $a_2 = 0$, and have to handle the term

$$n^{-1+q\alpha} \sum_{j,k=\lfloor ns \rfloor \vee n_0}^{\lfloor nt \rfloor - 1} \left| \mathbb{E} \left[\Delta W_{\frac{j}{n}} \Delta W_{\frac{k}{n}} \right] \right|^{q-a_3-a_4} \left| \mathbb{E} \left[\Delta W_{\frac{j}{n}} \Delta Y_{\frac{k}{n}} \right] \right|^{a_3} \left| \mathbb{E} \left[\Delta Y_{\frac{j}{n}} \Delta W_{\frac{k}{n}} \right] \right|^{a_4} \tag{4.9}$$

for all allowable values of a_3, a_4 with $a_3 + a_4 \geq 1$. Consider the decomposition

$$\begin{aligned}
& n^{-1+q\alpha} \sum_{j,k=\lfloor ns \rfloor \vee n_0}^{\lfloor nt \rfloor - 1} \left| \mathbb{E} \left[\Delta W_{\frac{j}{n}} \Delta W_{\frac{k}{n}} \right] \right|^{q-a_3-a_4} \left| \mathbb{E} \left[\Delta W_{\frac{j}{n}} \Delta Y_{\frac{k}{n}} \right] \right|^{a_3} \left| \mathbb{E} \left[\Delta Y_{\frac{j}{n}} \Delta W_{\frac{k}{n}} \right] \right|^{a_4} \\
& = n^{q\alpha-1} \sum_{j=\lfloor ns \rfloor \vee n_0}^{\lfloor nt \rfloor - 1} \left| \mathbb{E} \left[\Delta W_{\frac{j}{n}}^2 \right] \right|^{q-a_3-a_4} \left| \mathbb{E} \left[\Delta W_{\frac{j}{n}} \Delta Y_{\frac{j}{n}} \right] \right|^{a_3+a_4} \\
& + n^{q\alpha-1} \sum_{j=\lfloor ns \rfloor \vee n_0}^{\lfloor nt \rfloor - 1} \sum_{k=\lfloor ns \rfloor \vee n_0}^{j-1} \left| \mathbb{E} \left[\Delta W_{\frac{j}{n}} \Delta W_{\frac{k}{n}} \right] \right|^{q-a_3-a_4} \left| \mathbb{E} \left[\Delta W_{\frac{j}{n}} \Delta Y_{\frac{k}{n}} \right] \right|^{a_3} \left| \mathbb{E} \left[\Delta Y_{\frac{j}{n}} \Delta W_{\frac{k}{n}} \right] \right|^{a_4} \\
& + n^{q\alpha-1} \sum_{k=\lfloor ns \rfloor \vee n_0}^{\lfloor nt \rfloor - 1} \sum_{j=\lfloor ns \rfloor \vee n_0}^{k-1} \left| \mathbb{E} \left[\Delta W_{\frac{j}{n}} \Delta W_{\frac{k}{n}} \right] \right|^{q-a_3-a_4} \left| \mathbb{E} \left[\Delta W_{\frac{j}{n}} \Delta Y_{\frac{k}{n}} \right] \right|^{a_3} \left| \mathbb{E} \left[\Delta Y_{\frac{j}{n}} \Delta W_{\frac{k}{n}} \right] \right|^{a_4}.
\end{aligned}$$

We have, by Lemma 3 and Lemma 4,

$$\begin{aligned}
& n^{-1+q\alpha} \sum_{j,k=\lfloor ns \rfloor \vee n_0}^{\lfloor nt \rfloor - 1} \left| \mathbb{E} \left[\Delta W_{\frac{j}{n}} \Delta W_{\frac{k}{n}} \right] \right|^{q-a_3-a_4} \left| \mathbb{E} \left[\Delta W_{\frac{j}{n}} \Delta Y_{\frac{k}{n}} \right] \right|^{a_3} \left| \mathbb{E} \left[\Delta Y_{\frac{j}{n}} \Delta W_{\frac{k}{n}} \right] \right|^{a_4} \\
& \leq C n^{-1} \sum_{j=\lfloor ns \rfloor \vee n_0}^{\lfloor nt \rfloor - 1} j^{(a_3+a_4)(\alpha-2)} \\
& \quad + C n^{-1} \sum_{j=\lfloor ns \rfloor \vee n_0}^{\lfloor nt \rfloor - 1} j^{a_3(2H-2)-a_4\gamma} \sum_{k=\lfloor ns \rfloor \vee n_0}^{j-1} k^{-a_3\gamma+a_4(2H-2)} |j-k|^{(q-a_3-a_4)(\alpha-2)} \\
& \quad + C n^{-1} \sum_{k=\lfloor ns \rfloor \vee n_0}^{\lfloor nt \rfloor - 1} k^{-a_3\gamma+a_4(2H-2)} \sum_{j=\lfloor ns \rfloor \vee n_0}^{k-1} j^{a_3(2H-2)-a_4\gamma} |k-j|^{(q-a_3-a_4)(\alpha-2)} \\
& \leq C n^{-1} \left((\lfloor nt \rfloor - \lfloor ns \rfloor)^{[(a_3+a_4)(\alpha-2)+1] \vee 0} + (\lfloor nt \rfloor - \lfloor ns \rfloor)^{[q(\alpha-2)+2] \vee 0} \right. \\
& \quad \left. + (\lfloor nt \rfloor - \lfloor ns \rfloor)^{[a_3(2H-2)-a_4\gamma+1] \vee 0} + (\lfloor nt \rfloor - \lfloor ns \rfloor)^{[a_4(2H-2)-a_3\gamma+1] \vee 0} \right). \quad (4.10)
\end{aligned}$$

Then (4.8) and (4.10) imply (4.4) because $\alpha < 2 - \frac{1}{q}$.

Proof of (4.5): We have

$$G_n(t) - \tilde{G}_n(t) = n^{-\frac{1}{2}} \sum_{j=n_0}^{\lfloor nt \rfloor - 1} \left(\beta_{j,n}^{-q} - \left\| \Delta W_{\frac{j}{n}} \right\|_{L^2(\Omega)}^{-q} \right) I_q^W \left(\mathbf{1}_{\left[\frac{j}{n}, \frac{j+1}{n} \right)}^{\otimes q} \right)$$

and we can write, using (4.3) for any $j \geq n_0$,

$$\left| \beta_{j,n}^{-q} - \left\| \Delta W_{\frac{j}{n}} \right\|_{L^2(\Omega)}^{-q} \right| = (\kappa^{-1} n^\alpha)^{\frac{q}{2}} \left| (1 + \theta_{j,n})^{-\frac{q}{2}} - 1 \right| \leq C (\kappa^{-1} n^\alpha j^{\alpha-2})^{\frac{q}{2}}.$$

This leads to the estimate

$$\begin{aligned}
& \mathbb{E} \left[\left| G_n(t) - \tilde{G}_n(t) - (G_n(s) - \tilde{G}_n(s)) \right|^2 \right] \leq C n^{-1} \\
& \quad \times \left(\sum_{j=\lfloor ns \rfloor \vee n_0}^{\lfloor nt \rfloor - 1} j^{\alpha-2} + \sum_{j,k=\lfloor ns \rfloor \vee n_0, j \neq k}^{\lfloor nt \rfloor - 1} |j-k|^{q(\alpha-2)} \right) \\
& \leq C n^{-1} (\lfloor nt \rfloor - \lfloor ns \rfloor)^{(\alpha-1) \vee 0} + (\lfloor nt \rfloor - \lfloor ns \rfloor)^{[q(\alpha-2)+2] \vee 0},
\end{aligned}$$

which implies (4.5).

This concludes the proof of Theorem 2.

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